Challenging Courses and Curricula: A Model for All Students

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Abstract: The Common Core State Standards for Mathematical Practice describe "varieties of expertise that mathematics educators at all levels should seek to develop in their students." [CC] These eight mathematical practices are consistent with the definition for Challenging Courses and Curricula developed by the NSF-supported Greater Birmingham Mathematics Partnership (GBMP). The four main attributes of challenging mathematics courses are that they deepen knowledge of important mathematical ideas, promote inquiry and reflection, develop productive disposition, and foster communication. We describe classroom practice with examples from GBMP classrooms that illustrate our definition of challenging courses and develop the student proficiencies described in the Standards for Mathematical Practice.

Introduction

The Common Core State Standards (CCSS) for Mathematical Content and Practice have been adopted by 46 states and the District of Columbia. The CCSS Standards for Mathematical Content which describe the content to be taught at each grade have received the majority of the attention by many state boards of education, school districts, administrators and teachers. The CCSS Standards for Mathematical Practice which describe mathematical habits of mind such as sense-making, reasoning, perseverance, and communicating mathematical arguments are vitally important but have received much less attention. We have found the reasons for this include that: (1) teachers and administrators do not understand what some of the mathematical practices are trying to describe; (2) many teachers were taught in traditional lecture style and so have never experienced learning in an environment focused on developing the mathematical practices [KMD]; (3) teachers struggle to envision what classroom practice would look like where students learn content through engaging in mathematical practices; and (4) many administrators and teachers view covering content standards as the way to raise test scores and the practice standards as extras rather than viewing the mathematical practices as basic skills of the Common Core State Standards.

The Greater Birmingham Mathematics Partnership (GBMP) has been implementing a professional development model for seven years that promotes classroom instruction consistent with the Standards for Mathematical Practice. GBMP was guided in this model by our definition of Challenging Courses and Curricula.

When teachers and administrators refer to "challenging" mathematics courses, they are often referring to only the most advanced coursework available (such as a calculus course taken in high school) or to an accelerated track of courses (such as an algebra course taken in 8th grade if the majority of students take algebra in 9th grade). This conception of challenging courses also appears in the literature ([DOE], [NMAP]). We offer a different framework for challenging mathematics courses that asserts that all courses can and should be challenging for the students who take them and should result in students who develop expertise with the mathematical practices. We describe below the definition of challenging courses developed by the Greater Birmingham Mathematics Partnership (GBMP) with the support of the National Science Foundation Math Science Partnership program (award #EHR-0632522).

Challenging courses and curricula: (1) help students deepen their knowledge of the big ideas in mathematics; (2) promote student inquiry and reflection; (3) support the development of productive disposition; and (4) foster articulate written and oral communication. We also recognize that aligned assessment practices can positively impact these four overarching goals.

In our work, we have seen classrooms where students are highly engaged in solving complex mathematics tasks, where students make sense of the mathematics they are doing, and where talking mathematics is the norm. In these classrooms, all students are engaged but no student is held back from taking the mathematics as far as possible. In a challenging course, teachers think of mathematics as a sense-making discipline and help students make connections between and among seemingly unrelated mathematical ideas rather than viewing mathematics as sets of isolated skills and domains. The four components of challenging courses are evident in these classrooms. We describe below the classroom environment and instructional practices found in challenging courses.

Classroom Environment and Instructional Practices in Challenging Courses

1. Big Mathematical Ideas

In challenging courses, students investigate a coherent collection of problems organized around big mathematical ideas. Rather than teaching isolated skills on an accelerated timeline, we view challenging courses as going deeply into the mathematical study of a few big ideas. In short, we fully appreciate the seemingly contradictory notion that by teaching fewer mathematics topics, but teaching them more thoroughly, learners will come to understand more mathematics and understand it as a fabric of connected and related ideas.

For example, a collection of problems might revolve around developing an understanding of what a fraction is by investigating various models for fractions such as area, linear, and set models and confronting fractions in a variety of contexts such as money, quantities, time, and distance. Students develop an understanding of 1/3 as a quantity just as they have a grasp of 3 as a quantity.

To illustrate, on one visit to a challenging classroom, we observed the teacher starting with a whole group activity about what a fraction is, followed by whole class discussion to bring out the major ideas. After the class discussion, students investigate a collection of fraction problems posted around the room that allow them to confront these mathematical ideas in multiple contexts. This is often called a “menu” of problems where students have a choice as to the order in which to tackle the problems, how long to work on each problem, and whether to work independently or in small groups. One menu problem uses Cuisenaire rods as a model for fractions (linear model) and other problems involve color counters (set model), pattern blocks (area model), etc. During menu time, the teacher is interacting with students and asking questions to reveal any ‘soft spots’ in their understanding and help them to construct deeper level understandings. An important part of the menu structure is time for whole class processing of selected tasks, described below under “Communication.”

Also central to our conception of challenging courses is the belief that all learners are capable of having powerful mathematical ideas. It is important for the instructor to meet a wide range of learner needs while challenging every learner. The tasks are designed so that all students have access to the mathematics while not limiting anyone’s thinking. “Dessert” problems are optional menu tasks to challenge even the most mathematically sophisticated student.

GBMP recognizes that practice with new skills and concepts is essential if students are to learn how to put mathematics to work in empowering ways. In challenging courses, such practice is provided within engaging and mathematically important contexts that also serve to build more productive mathematical dispositions.

2. Inquiry and Reflection

GBMP’s conception of challenging courses is based on the belief that coming to know and understand important mathematical ideas takes time and that learning occurs through a process of inquiry and reflection. We view confusion—the cognitive dissonance that accompanies “not knowing”—as a natural and even desirable part of the process of constructing new understandings. Challenging courses provide opportunities for students to struggle with problems, to find their own ways of solving them, and to recognize that there is usually not just one way to solve a problem. The dilemma for teachers is that they were often taught that a teacher’s job is to help or teach by giving clear explanations of how to best solve problems. We have learned, however, that this natural inclination to want to put confusion to rest, and to “help” those who are struggling, is often counterproductive when it comes to developing mathematical understandings and productive dispositions.

We want to clarify our use of the word “confusion” and not leave the impression that we view all confusion as desirable. Some kinds of confusion need to be cleared up (e.g., when some kind of “social knowledge” such as how a symbol is used or how a problem is posed has not been made clear). We have come to believe that teaching by telling rarely leads to deep mathematical understandings or productive mathematical dispositions. When students ask for help, instructors interact with them in ways that do not direct their thinking. Rather than solving a problem for a group or individual, instructors listen to students’ thinking and ask probing questions to help students find their own ways through the problems and honor their struggles.

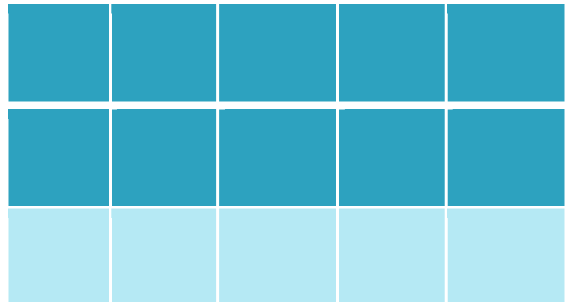
To illustrate, we observed students investigating the following problem.

> The Square Dance Problem: For the first dance at the school square dance, 2/3 of the boys danced with 3/5 of the girls. What fraction of the students were dancing?

We recommend that you stop and think about this problem before reading on. Students worked in small groups using color tiles to represent and make sense of the problem. Initially, one group thought they had a solution, but it involved finding a common denominator. They confronted the confusion that 9/15 + 10/15 = 19/15 is more than 100% of the students. Another group created the following diagram and said that 3/5 of the girls are dancing with 2/3 of the boys so 19/30 of the students are dancing.

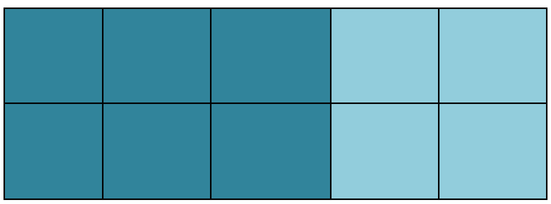
3/5 of the girls are dancing

2/3 of the boys are dancing

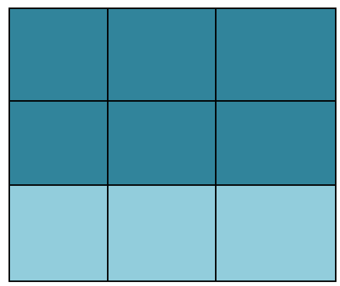


They confronted the confusion that one dancing boy did not have a partner. Eventually these groups wrestled their way out of their confusion and found a geometric solution that made sense to them. Using the diagram below, they argued that 3/5 of the girls were dancing with 2/3 of the boys, so 12/19 of all the students were dancing.

3/5 of the girls are dancing



2/3 of the boys are dancing



Another group approached the problem algebraically and reasoned that $\frac{2}{3}B = \frac{3}{5}G$ where *B* is the number of boys and *G* is the number of girls. This group faced confusion about what to do next and made several unsuccessful attempts, eventually reasoning their way to the following solution that made sense to them. Since $\frac{2}{3}B = \frac{3}{5}G$, the number of boys is 9/10 times the number of girls, $B = \frac{9}{10}G$. Therefore, the fraction of students dancing is $\dfrac{\frac{2}{3}B + \frac{3}{5}G}{B+G} = \dfrac{\frac{3}{5}G + \frac{3}{5}G}{\frac{9}{10}G + G} = \dfrac{\frac{6}{5}G}{\frac{19}{10}G} = \frac{12}{19}$.

3. Productive disposition

Challenging courses are designed with the understanding that learning mathematics involves hard work. Even students who are confident in their mathematical content knowledge often encounter disequilibrium when they are asked to see problems in multiple ways or to solve a problem where the solution path is not immediately obvious to them. All participants, no matter their level of competence or confidence with mathematics, are engaged with mathematical tasks that demand perseverance. Participants learn what it means to struggle and to experience the exhilaration of finally solving a problem or understanding a mathematical idea. Students come to know that the degree of exhilaration or joy they experience in solving a problem is often directly proportional to the amount of struggle and effort expended.

Challenging courses foster a productive and supportive learning community. Participants come to care about each other's learning. They learn that in trying to understand the thinking of others, they understand mathematics at a deeper level themselves. Instructors make ongoing decisions throughout the course with the goal of developing autonomous learners. Participants learn how to ask for help by seeking guidance but not answers, and they learn how to help other students without doing the mathematical thinking for them. Rather than rescuing students, instructors interact with students in ways that build more powerful mathematical understandings and dispositions that diminish the need for future rescue.

As an example, we visited a third grade classroom in which students were exploring whether halving and doubling was a strategy that would always work for multiplication. Students had noticed that to find the answer for a multiplication problem, you could halve one factor and double the other factor and it would still give the same product (e.g. $5 \times 18 = 10 \times 9 = 90$). One student said that this strategy was good for working with even numbers, but it wouldn't work with two odd numbers. Another student said that if the strategy was going to work, it would have to work in all cases, so let's see if it works with 7×7. The teacher knew that this would be a messy problem, but instead of stopping the students or suggesting an easier problem, she encouraged them to give it a try.

Alethia: $7 \times 7 = 49$; double 7 to get 14, and what's half of 7?

Mark: You can halve 6 to get 3 and half of 1 is ½, so half of 7 is 3½ .

Shandra: So how do we multiply 3 ½ × 14?

Alethia: $3 \times 14 = 42$, and half of 14 is 7, and $42 + 7 = 49$.

Students: It works! Let's see if we can do it again!

Undaunted, the students proceeded to investigate the problem by halving 3 ½ and doubling 14 (1¾ × 28). The students reasoned their way through this by computing $1 \times 28 = 28$; ½ × 28 = 14; ¼ × 28 = 7, and 28 + 14 +7 = 49, which lead to cheers and applause at their own effort.

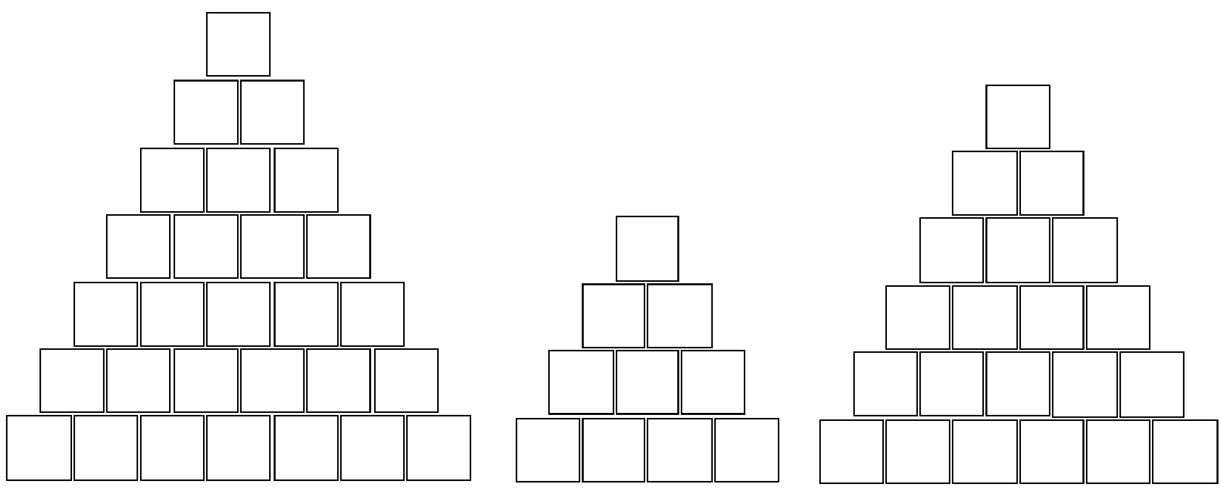
This vignette illustrates that the teacher valued investigation of mathematical ideas and believed students were capable of solving difficult mathematical problems. These 3rd graders believed that mathematics is supposed to make sense and they persisted in their sense-making process. They knew from experience that rich mathematical problems rarely have instant answers and so they were willing to persevere in reasoning through a challenging and unfamiliar problem.

4. Communication

Talking and writing mathematics is the norm in challenging courses. Communication of mathematical thinking occurs in small groups as students work together to make sense of the problems and during whole class processing. An essential element of whole class processing is establishing a safe environment in which all students and mathematical ideas are treated with respect. During processing, students volunteer to share their diverse ways of seeing and solving problems. As different solutions and various representations (geometric, verbal, numerical, algebraic) emerge, students deepen their understanding by making connections among various representations and solution paths. Whole class processing is done with an eye toward clarifying the mathematics involved and learning to consider, value, question, and build upon each others' mathematical ideas.

To illustrate, we observe a class processing the following problem.

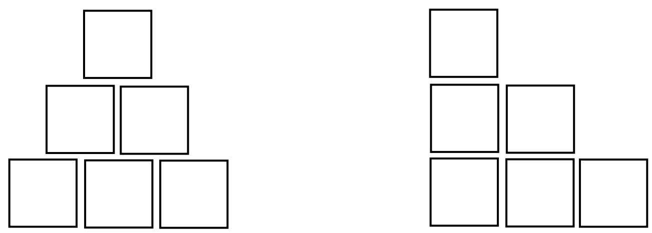
> The Building Problem: A few stages of an increasing pattern are shown below. How many tiles would it take to build Stage 10? What about any stage? (This problem is adapted from *Developing Number Concepts Using Unifix Cubes by* Kathy Richardson.)



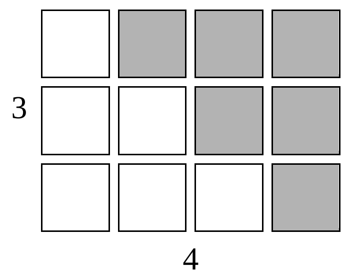
Stage 7 Stage 4 Stage 6

The teacher asks for volunteers and Patricia's hand goes up.

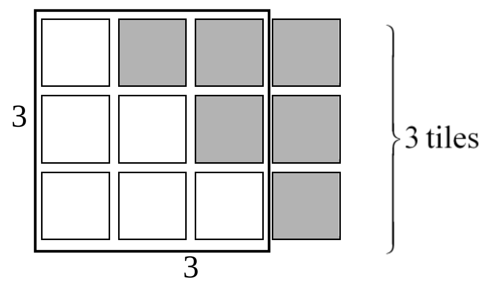
Patricia: I was building stage 3, moving tiles around, and I realized I could "left justify" stage 3 to look like this (the diagram on the right below).



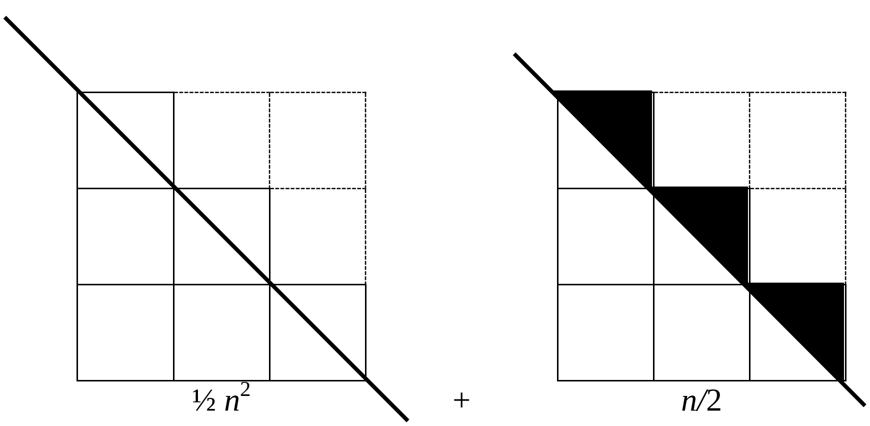
Then I put two copies of stage 3 together like this [see below]. Now it's easy to count that there are 3 × 4 tiles in all, but that's twice as many as I wanted, so there's really only (3 × 4)/2 tiles in stage three. For stage *n* there would be [*n* × (*n*+1)]/2 tiles.



Xavier chimes in that he built the same arrangement of tiles as Patricia, but he saw a 3×3 square plus 3 more tiles. Then he also divided by 2. For stage *n*, his formula was $\frac{n^2+n}{2}$.



Next JaMichal volunteers that he solved the problem by completing a square with color tiles, dividing the square in half, and adding back half of each tile on the diagonal for a result of ½ n^2 + n/2.



Conclusion

The framework above describes a broadly applicable vision for challenging mathematics courses. Whereas the common interpretation of "challenging" mathematics is relevant only for a small population of students enrolled in accelerated classes or enrichment programs, this definition applies to all mathematics courses and all students. The universality of the definition was one aim of the design—it is applicable not only to K-12 but also to undergraduate and graduate courses and professional development institutes. But it is also universal in another sense. Under this interpretation, challenging courses help students develop mathematical practices that transcend any particular mathematics course; it builds their capacity to learn as much as it builds their knowledge of arithmetic, or geometry, or differential equations. The broad adoption of the CCSS represents a unique opportunity to shift mathematics instruction not only toward more focused and coherent content standards but also toward students learning content through engaging in the mathematical practices so that all students experience challenging courses.

This framework was developed by a partnership of nine demographically diverse school districts, a large research university, a small liberal arts college, and an educational nonprofit organization, and there was consensus across all levels about the operational definition.* The partnership is not arguing against offering advanced courses, but rather advocating that every course should provide a challenging learning environment. In elaborating on the Equity Principle [NCTM], the NCTM states that "all students need access each year they are in school to a coherent, challenging mathematics curriculum." This GBMP framework supports this principle and is summarized in our Operational Definition of Challenging Courses and Curriculum below.

Operational Definition of Challenging Courses and Curricula

1. Big Mathematical Ideas
   - Teach for understanding. This refers to helping students achieve “an integrated and functional grasp of mathematical ideas.”([NRC]) This includes developing conceptual understanding, strategic competence, and procedural fluency.
   - Introduce a mathematical idea by posing problems that motivate it.
   - Provide a coherent collection of problems organized around a big mathematical idea.
   - Provide opportunities for students to use multiple representations of a mathematical idea.
   - Provide opportunities for students to explore real-world problems connected to big mathematical ideas.

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*In the process of developing this definition of challenging courses and curricula, GBMP drew on the National Research Council’s description of the “intertwined strands of proficiency” in *Adding It Up* [NRC]. We also made use of the “teaching for understanding: guiding principles” articulated in the California State Department of Education *Mathematics: Model Curriculum Guide* [CA] as well as other sources ([NRC2], [NRC3], [WeissHQ], [WeissLIC], [NCSM], [Polya], [Bowen], [Parker], and [Parker2]). We also drew on the expertise of the GBMP National Advisory Board which includes recognized experts in mathematics, education, and assessment.

2. Inquiry and Reflection
    - Engage students in inquiry.
    - Communicate that learning mathematics should be a sense-making process.
    - Ask students to justify their thinking.
    - Ask students to engage in reflection.
    - Encourage students to think critically about mathematical ideas and solutions.
    - Encourage diverse ways of thinking.
    - Communicate that both accuracy and efficiency are important.
    - Incorporate technology when appropriate.

3. Productive Disposition
    - Help students develop persistence, resourcefulness and confidence.
    - Help students become autonomous learners.
    - Provide a safe, respectful learning environment.

4. Communication
    - Promote the development of mathematical language.
    - Value written communication by asking students to explain their ideas in writing.
    - Value verbal communication by asking individuals and groups to articulate their thinking.
    - Value the role of communication in developing intellectual community in the classroom.
    - Establish clear expectations for mathematical assignments.

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