Spectral and Dynamical Properties of Non-Relativistic Matter Coupled to Quantized Radiation

Marcel Griesemer, UAB

Summerschool on Large Coulomb Systems - QED
Nordfjordeid (Norway), August 11 – 18, 2003.

Collaborators:

Jürg Fröhlich, Benjamin Schlein (FGS)
Elliott Lieb, Michael Loss

Acknowledgment:

Work in part supported by the U.S. National Science Foundation under Grant DMS 0100160.

1. Atoms and Molecules at Energies below the Ionization Threshold

- Ionization threshold and localization of bound states
- Existence of a ground state
- Existence of photon scattering states
- Asymptotic completeness of Rayleigh scattering
- Relaxation to the ground state

2. Compton Scattering

Hilbert space

$$\mathcal{H} = \mathcal{H}_{el} \otimes \mathcal{F} \qquad \mathcal{H}_{el} = \wedge_{i=1}^{N} L^2(\mathbb{R}^3; \mathbb{C}^2), \quad \mathcal{F} = \oplus_{n\geq 0} \otimes_s^n L^2(\mathbb{R}^3; \mathbb{C}^2)$$

Hamiltonian

$$H_N = \sum_{i=1}^{N} (p_i + \sqrt{\alpha} A(x_i))^2 + \sqrt{\alpha}\frac{g}{2}\sigma_i \cdot B(x_i) + V + I + H_f$$

$$A(x) = \sum_{\lambda=1,2} \int_{|k|\leq\Lambda} \frac{\varepsilon_\lambda(k)}{|k|^{1/2}} \left(e^{ik\cdot x} a_\lambda(k) + e^{-ik\cdot x} a_\lambda^*(k)\right) dk, \qquad B = \text{curl} A$$

$$H_f = \sum_\lambda \int \omega(k) a_\lambda^*(k) a_\lambda(k)\, dk, \qquad N_f = \sum_\lambda \int a_\lambda^*(k) a_\lambda(k)\, dk$$

$g \in \mathbb{R}$, $\Lambda, \alpha \in \mathbb{R}_+$ are arbitrary, unless stated otherwise

V, I are real-valued and symmetric with respect to permutation of the particle coordinates. Furthermore

$$V, I \in L^2_{loc}(\mathbb{R}^{3N}), \qquad V_-, I_- \ll -\Delta.$$

H_N is symmetric and bounded from below on a suitable dense subspace $D \subset \mathcal{H}$. We define a self-adjoint Hamiltonian by the Friedrichs' extension of H_N. (Hiroshima: H_N is self-adjoint on $D(-\Delta + H_f)$ under suitable assumptions on V, I.)

Example: $X = (x_1, \ldots, x_N) \in \mathbb{R}^{3N}$,

$$V(X) = -\sum_{i=1}^{N} \sum_{j=1}^{K} \frac{Z_j \alpha}{|x_i - R_j|}, \qquad I(X) = \sum_{i<j} \frac{\alpha}{|x_i - x_j|}.$$

CREATION- AND ANNIHILATION OPERATORS

Let $h \in L^2(\mathbb{R}^3; \mathbb{C}^2)$, $\varphi = (\varphi_0, \varphi_1, \ldots) \in \mathcal{F}_{fin}$.

$$[a^*(h)\varphi]_n := \sqrt{n}\mathcal{S}_n[h \otimes \varphi_{n-1}]$$

Canonical commutation relations:

$$[a(h), a^*(g)] = (h, g), \qquad [a^\#(h), a^\#(g)] = 0.$$

Formally,

$$a^*(h) = \sum_{\lambda=1,2} \int h_\lambda(k) a^*_\lambda(k)\, dk.$$

Example:

$$A_i(x) = a(G_{x,i}) + a^*(G_{x,i}), \qquad G_{x,i}(k,\lambda) = \chi_\Lambda(k) \frac{\varepsilon_\lambda(k)_i}{\sqrt{|k|}} e^{-ik\cdot x}.$$

IONIZATION THRESHOLD

$$\Sigma = \lim_{R\to\infty} \Sigma_R, \qquad \Sigma_R = \inf_{\varphi\in D_R,\, \|\varphi\|=1} \langle \varphi, H_N \varphi \rangle$$

$$D_R = \{\varphi \in D : \varphi(X) = 0 \text{ if } |X| < R\}$$

Here $|X|^2 = \sum_{i=1}^N x_i^2$ and $\varphi(X) \in \mathbb{C}^{(2^N)} \otimes \mathcal{F}$.

For atoms and molecules $\Sigma = \min_{N'\geq 1}(E^V_{N-N'} + E^0_{N'})$ where $E^V_N = \inf \sigma(H_N)$.

Remark. If $H_N = -\Delta + V$ then $\Sigma = \inf \sigma_{ess}(H_N)$, (A. Persson, 1960)

Theorem (BFS '98, G. '02) If $\lambda, \beta \in \mathbb{R}$ such that $\lambda + \beta^2 < \Sigma$ then

$$\|(e^{\beta|X|} \otimes 1) E_\lambda(H_N)\| < \infty.$$

That is, if $\varphi \in \operatorname{Ran} E_\lambda(H)$,

$$\int e^{2\beta|X|} \|\varphi(X)\|^2 dX \leq C \|\varphi\|^2,$$

This theorem is central for the following! Assumptions on H_N: $f D(H_N) \subset D(H_N)$, $\langle f\varphi, H_N f\varphi \rangle \leq a\langle \varphi, H\varphi \rangle + b\langle \varphi, \varphi \rangle$, and

$$[f, [f, H_N]] = -2|\nabla f|^2$$

for all $f \in C^\infty(\mathbb{R}^{3N}), f, \nabla f \in L^\infty$.

EXISTENCE OF A GROUND STATE

Arai, BFS, Gérard, Hiroshima, Hirokawa, Spohn.

Theorem (Lieb, Loss, G. 2001)

If $E_N \equiv \inf \sigma(H_N) < \Sigma$ then E_N is an eigenvalue of H_N.

It remains a variational problem: find $\psi \in \mathcal{H}$ such that $\langle \psi, H\psi \rangle < \Sigma \langle \psi, \psi \rangle$. More convenient characterization of Σ:

Theorem (Lieb, Loss, G. 2001, 2002)

If $V(X) = \sum_i v(x_i)$, $I(X) = \sum_{i<j} w(x_i - x_j)$ and

$$v(x) \to 0, \quad w(x) \to 0 \quad \text{as } |x| \to \infty,$$

then

$$\Sigma = \min_{N' \geq 1} \left(E^V_{N-N'} + E^0_{N'} \right).$$

Example for $N = 1$. By a simple variational argument

$$E_1^V \leq E_1^0 + \inf \sigma(-\Delta + V).$$

If $\lim_{|x|\to\infty} V(x) = 0$ and $\inf \sigma(-\Delta + V) < 0$, it follows that $E_1^0 = \Sigma$ and $E_1^V < \Sigma$.

Enhanced binding $E_1^V < E_1^0 + \inf \sigma(-\Delta + V)$? Yes! Hiroshima, Spohn, Hainzl, Catto, Seiringer, Vugalter, Vougalter, Chen.

Theorem (Lieb, Loss 2003)
For atoms and molecules with $N - 1 < Z = \sum_j Z_j$.

$$E_N^Z < \min_{N' \geq 1} (E_{N-N'}^Z + E_{N'}^0).$$

Previous: BFS / Barbaroux, Chen, Vugalter

ELEMENTS OF THE PROOF

1. induction in N. Can assume $E^V_{N-N'}$ is an eigenvalue, i.e., the electrons are localized. Find approximate ground state $\psi^V_{N-N'}$ in which the photons are localized as well. Prize in energy at most $o(R^{-1})$, $R =$ localization length.

2. Find approximate ground state $\psi^0_{N'}$ for $H^0_{N'}$ where electrons and photons are localized. Prize $o(R^{-1})$.

3. merge $\psi^V_{N-N'}$ and $\psi^0_{N'}$ to an N-electron state ψ_N with

$$\langle \psi_N, H_N \psi_N \rangle \leq E^V_{N-N'} + E^0_{N'} - c/R + o(R^{-1}).$$

Lemma $E_1^V \leq E_1^0 + \inf \sigma(-\Delta + V)$.

Proof. Let $e_0 = \inf \sigma(-\Delta + V)$. Choose χ, ϕ, $\|\chi\| = 1$, $\|\phi\| = 1$:

$$\langle \chi, (-\Delta + V)\chi \rangle < e_0 + \varepsilon/2, \qquad \langle \phi, H_1^0 \phi \rangle < E_1^0 + \varepsilon/2.$$

Let

$$\psi_y(x) = \chi(x)\phi_y(x), \qquad \phi_y = e^{-i(p+P_f)\cdot y}\phi.$$

And show that

$$\int \langle \psi_y, (H_1^V - [E_1^0 + e_0 + \varepsilon])\psi_y \rangle dy < 0.$$

using

$$\chi(p + A)^2 \chi = (p + A)\chi^2(p + A) + \chi(-\Delta\chi).$$

It follows that $\langle \psi_y, (H_1^V - [E_1^0 + e_0 + \varepsilon])\psi_y \rangle < 0$ for some y. $\square$

ASYMPTOTIC COMPLETENESS

Assume $\inf \sigma(H) < \Sigma =$ ionization threshold. Let

$$\mathcal{H}_\Sigma := \mathrm{Ran}\, E_{(-\infty,\Sigma)}(H) \neq \{0\}$$

Electrons in $\varphi \in \mathcal{H}_\Sigma$ are localized near origin.

Expect: Given $\varphi \in \mathrm{Ran}\, E_\lambda(H)$

$$e^{-iHt}\varphi \overset{t\to\infty}{\simeq} \text{ superposition of states of the form}$$

$$a^*(h_{1,t}) \ldots a^*(h_{n,t}) e^{-iEt} \varphi_E \tag{1}$$

$$h_{i,t}(k) := e^{-i\omega(k)t} h_i(k), \qquad H\varphi_E = E\varphi_E, \qquad E := \inf \sigma(H).$$

Can only be true if E is the only eigenvalue of H below Σ. Otherwise φ_E may be any eigenstate with energy $E < \Sigma$.

Define:

$$a_+^*(h) = s - \lim_{t\to\infty} e^{iHt} a^*(h_t) e^{-iHt}$$

(if existent), then (1) says

$$\varphi \in \overline{\text{span}}\{a_+^*(h_1)\dots a_+^*(h_n)\varphi_E | n \in \mathbb{N},\ h_i \in L^2, E < \Sigma,\ H\varphi_E = E\varphi_E\}$$
$$=: \mathcal{H}_+, \qquad \text{(scattering states)}$$

Asymptotic completeness of Rayleigh scattering

$$\boxed{\mathcal{H}_\Sigma \subset \mathcal{H}_+}$$

Note that:

$$e^{-iHt} a_+^*(h_1)\dots a_+^*(h_n)\varphi_E \overset{t\to\infty}{\simeq} a^*(h_{1,t})\dots a^*(h_{n,t}) e^{-iEt}\varphi_E$$

Remarks.

- Existence of scattering states well understood

  (Høgh-Krohn, FGS)

- Asymptotic completeness known for systems with

  - massive photons and $\Sigma = \infty$

    (Hübner, Spohn / Dereziński, Gérard)

  - massive photons or infrared cutoff interaction, $\Sigma < \infty$

    (FGS)

  - (perturbations of) explicitly soluble models (Arai / Spohn)

EXISTENCE OF SCATTERING STATES

For simplicity

$$H = p^2 + V + H_f + \phi(G_x) \qquad G_x(k) = \frac{\kappa(k)}{\sqrt{\omega(k)}} e^{-ik\cdot x}$$

$\kappa \in C_0^\infty(\mathbb{R}^3),\ \omega(k) = \sqrt{k^2 + m^2},\ m \geq 0.$ We ask whether

$$a_+^*(h)\varphi = \lim_{t\to\infty} e^{iHt} a^*(h_t) e^{-iHt}\varphi =: \lim_{t\to\infty} \varphi(t)$$

exists for given $h \in C_0^\infty(\mathbb{R}^3 \backslash \{0\})$, $\varphi \in D(H)$.

Sufficient condition: (Cook's argument)

$$\int_1^\infty \left\| \frac{d}{dt}\varphi(t) \right\| dt < \infty$$

$$\varphi(t) = e^{iHt} a^*(h_t) e^{-iHt}\varphi = e^{iHt} \underbrace{e^{-iH_f t} a^*(h) e^{iH_f t}}_{a^*(h_t)} e^{-iHt}\varphi$$

$$\varphi'(t) = e^{iHt}\, i[H - H_f, a^*(h_t)]\, e^{-iHt}\varphi = e^{iHt}\, i(G_x, h_t)\, e^{-iHt}\varphi$$

$$\|\varphi'(t)\| \le \|(G_x, h_t) e^{-iHt}\varphi\| \le \sup_{x\in\mathbb{R}^3} |(G_x, h_t)|\, \|\varphi\|.$$

$$(G_x, h_t) = \int e^{i(k\cdot x - \omega t)} \underbrace{\kappa(k)\omega(k)^{-1/2} h(k)}_{\in C_0^\infty(\mathbb{R}^3\setminus\{0\})} d^3k \qquad (2)$$

$$\sup_x |(G_x, h_t)| \le \text{const} \begin{cases} t^{-3/2} & m > 0 \\ t^{-1} & m = 0 \end{cases}$$

$m > 0$: $a_+^*(h)$ exists. $m = 0$: need better estimate.

The phase in (2) is stationary for $|x/t| = |\nabla\omega| = 1$. We have

$$\sup_{x:||x/t|-1|\geq\varepsilon} |(G_x, h_t)| \leq \frac{C_n}{t^n}, \qquad n \in \mathbb{N}$$

Remains to estimate

$$\int_1^\infty \frac{dt}{t} \|\chi_{[1-\varepsilon,1+\varepsilon]}(|x|/t)e^{-iHt}\varphi\|$$

Suppose $\varphi \in \operatorname{Ran} E_\lambda(H)$, $\lambda < \Sigma$, $\lambda + \beta^2 < \Sigma$.

$$\|\chi_{[1-\varepsilon,1+\varepsilon]}(|x|/t)e^{-\beta|x|}e^{\beta|x|}e^{-iHt}\varphi\| \leq \text{const } e^{-\beta(1-\varepsilon)t}.$$

Propagation estimate for relativistic electrons

$$H = \sqrt{p^2+1} + V + \phi(G_x)$$

Theorem (FGS 2001) Suppose $f \in C_0^\infty(\mathbb{R})$ and $\varepsilon > 0$ is small enough. Then, for $\mu > 1/2$

$$\int_1^\infty \frac{dt}{t^\mu} \|\chi_{[1-\varepsilon,\infty)}(|x|/t)e^{-iHt}f(H)\varphi\|^2 \leq C_\mu \|\langle x\rangle^{1/2} f(H)\varphi\|^2$$

Theorem (FGS, 2000) If $\int |h_i(k)|^2(1+|k|^{-1})dk < \infty$, $i = 1\ldots,n$, and $\varphi \in D((H+i)^{n/2})$, then

$$a_+^*(h_1)\ldots a_+^*(h_n)\varphi = \lim_{t\to\infty} e^{iHt}a^*(h_{1,t})\ldots a^*(h_{n,t})e^{-iHt}\varphi$$

Non-relativistic electrons

Using

$$a_+^*(h)\,\mathrm{Ran}\,E_\lambda(H) = \mathrm{Ran}\,E_{\lambda+M}(H)$$

if $M := \sup\{\omega(k)\,|\,h(k) \neq 0\}$, one can prove:

Theorem (FGS) Suppose $\varphi \in E_\lambda(H)\mathcal{H}$, $h_i \in L^2(\mathbb{R}^3; (1+|k|^{-1})dk)$, $M_i := \sup\{\omega(k)|h_i(k) \neq 0\}$, and

$$\lambda + \sum_{i=1}^{n} M_i < \Sigma.$$

Then

$$a_+^*(h_1)\dots a_+^*(h_n)\varphi = \lim_{t\to\infty} e^{iHt}a^*(h_{1,t})\dots a^*(h_{n,t})e^{-iHt}\varphi.$$

Theorem (Dereziński-Gérard, FGS)

Suppose $H = p^2 + V + H_f + g\phi(G_x)$, $m > 0$ and

$$\omega(k) = \sqrt{k^2 + m^2}$$

or

$$\omega(k) = |k| \quad \text{and} \quad G_x(k) = 0 \quad \text{if } |k| \leq m$$

Then

$$\text{Ran } E_{(-\infty,\Sigma)}(H) \subset \mathcal{H}_+.$$

Remark. If $\omega(k) = |k|$ (+IR-cutoff), $\mu < \inf \sigma_{ess}(p^2 + V)$ then, for g small enough, $\inf \sigma(H_g)$ is the only eigenvalue of H_g and it is simple!

asymptotic completeness $\Rightarrow$ relaxation to the ground state.

RELAXATION TO THE GROUND STATE

Let φ_g be the unique, normalized ground state of H_g. Let $\mathcal{A}$ be the C^*-algebra generated by all operators

$$B \otimes e^{i\phi(h)}, \qquad B \in \mathcal{L}(\mathcal{H}_{el}),\ \phi(h) = a(h) + a^*(h),$$

with $h \in C_0^\infty(\mathbb{R}^3)$.

Corollary (FGS) Suppose $\inf \sigma(H_g)$ is the only eigenvalue of H_g below μ and it is simple (satisfied for $|g|$ small & IR-cutoff, BFS/FGS). Then for all $\psi \in \operatorname{Ran} E_{(-\infty,\mu)}(H_g)$, $\|\psi\| = 1$

$$\lim_{t\to\infty} \langle \psi_t, A\psi_t \rangle = \langle \varphi_g, A\varphi_g \rangle, \qquad \text{for all } A \in \mathcal{A}$$

Another characterization of AC ($\Sigma = \infty$)

The asymptotic creation and annihilation operators $a_+^*(h)$, $a_+(h)$ satisfy the CCR:

$$[a_+(g), a_+^*(h)] = (g, h), \qquad [a_+^\#(g), a_+^\#(h)].$$

Let

$$\mathcal{H}_{vac} = \{\varphi \in D(H) |\, a_+(h)\varphi = 0 \text{ all } h \in L^2((1 + |k|^{-1})dk)\}$$

Then $\mathcal{H}_{vac} \supset \mathcal{H}_{pp}$ and if $m > 0$ then

$$\mathcal{H} = \overline{\text{span}}\{a_+^*(h_1) \cdots a_+^*(h_n)\varphi |\, \varphi \in \mathcal{H}_{vac}\}$$

(Hoegh-Krohn). Hence

$$\boxed{AC \Leftrightarrow \mathcal{H}_{vac} = \mathcal{H}_{pp}}$$

Remark. Does not make the proof easier.

FREE ELECTRON

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F} = L^2(\mathbb{R}^3, dx; \mathcal{F})$$

$$H_g = \Omega(p) + H_f + g\phi(G_x).$$

$\Omega(p) = \sqrt{p^2 + M^2}$ or $\Omega(p) = p^2/(2M)$, $p = -i\nabla$. $G_x(k) = e^{-ik\cdot x}\kappa(k)$. H commutes with the total momentum $p + P_f$.

$$U : L^2(\mathbb{R}^3, dx; \mathcal{F}) \longrightarrow L^2(\mathbb{R}^3, dP; \mathcal{F})$$

$$(U\varphi)_n(P, k_1, \ldots, k_n) = \widehat{\varphi}(P - \sum_{i=1}^{n} k_i, k_1, \ldots, k_n)$$

Then

$$(UHU^*)(P) = H(P)\varphi(P) \qquad \varphi(P) \in \mathcal{F}$$

$$H_g(P) = \Omega(P - P_f) + H_f + g\phi(\kappa)$$

The spectrum if $g = 0$

$$H_{g=0}(P)|vac\rangle = \Omega(p)|vac\rangle$$

$$E_0(P) = \inf \sigma(H_0(P)) = \min \left(\Omega(P), \inf_k [\Omega(P-k) + |k|]\right)$$

If $|\nabla\Omega(P)| \leq 1$ then $\Omega(P-k) \geq \Omega(P) - |k|$ and hence

$$\boxed{E_0(P) = \Omega(P)}$$

(picture)

Let $\mathcal{F}_\sigma = \oplus_{n\geq 0} \otimes_s^n L^2(|k| \geq \sigma)$, $|\nabla\Omega(P)| \leq \beta < 1$.

$$\inf \sigma(H_0(P)\restriction \mathcal{F}_\sigma) = \Omega(P)$$

$$\inf \sigma_{\text{ess}}(H_0(P)\restriction \mathcal{F}_\sigma) \geq \Omega(P) + (1-\beta)\sigma,$$

gap! Suppose $\kappa \in C^\infty(\mathbb{R}^3)$ and

$$\boxed{\kappa(k) = 0 \text{ if } |k| \leq \sigma, \ \sigma > 0}$$

Theorem. (Fröhlich '73) Let $\sigma > 0$, $g \in \mathbb{R}$. If

$$(a) \quad \Omega(P) = \sqrt{P^2 + M^2} \quad \text{or}$$

$$(b) \quad \Omega(P) = P^2/(2M) \quad \text{and} \quad |P|/M \leq \sqrt{3} - 1$$

then $E_g(P) = \inf \sigma(H_g(P))$ is an eigenvalue of $H_g(P)$.

Remark. The eigenstate ψ_P belonging to $E_g(P)$ is called *dressed one-electron state* (DES).

ABSENCE OF EXCITED DES

Theorem. (FGS 2003) Let $\Sigma \in \mathbb{R}$. Then there exists $g_\Sigma > 0$ such that, for all $g : |g| \leq g_\Sigma$

$$\sigma_{pp}(H_g(P)) \cap (E_g(P), \Sigma] = \emptyset.$$

Proof (Sketch). Positive commutator + virial theorem: Let $A = \mathrm{d}\Gamma(a)$, $a = 1/2(\widehat{k} \cdot y + y \cdot \widehat{k})$, $y = i\nabla_k$. Then

$$[iH_g(P), A] = N_f - \nabla\Omega(P - P_f) \cdot \mathrm{d}\Gamma(\widehat{k}) - g\phi(ia\kappa).$$

If $\varphi \in \mathrm{Ran}\, E_\Sigma(H_g(P)) \cap D(N)$, $H_g(P)\varphi = E\varphi$, then

$$\begin{aligned} 0 = \langle \varphi, [iH_g(P), A]\varphi \rangle &\geq \delta_\Sigma \langle \varphi, N_f \varphi \rangle - C|g| \\ &\geq \delta_\Sigma \langle \varphi, (1 - P_\Omega)\varphi \rangle - C|g| \end{aligned}$$

Choose $\langle \varphi, \Omega \rangle \geq 0$, then

$$\langle \varphi, (1 - P_\Omega)\varphi \rangle \geq \frac{1}{2} \|\varphi - \Omega\|^2.$$

It follows that

$$\|\varphi - \Omega\| \leq \left(\frac{2|g| C_\Sigma}{\delta_\Sigma} \right)^{1/2}$$

Hence, for $|g|$ small enough, $\dim(\operatorname{Ran} E_\Sigma(H_g(P)) \cap \mathcal{H}_{pp}) = 1$. $\square$

Dressed one-electron wave packets

$$\mathcal{H}_{\text{des}} = \{\varphi \in \mathcal{H} | \, U\varphi(P) \in \text{span}\{\psi_P\}\}$$

Given $f \in L^2(\mathbb{R}^3, dP)$ let ψ_f be defined by $U\psi_f(P) = f(P)\psi_P$. Then

$$e^{-iHt}\psi_f = \psi_{f_t}, \qquad f_t(P) = e^{-iE(P)t} f(P).$$

We expect that, for any given $\varphi \in \mathcal{H}$,

$$e^{-iHt}\varphi \overset{t\to\infty}{\simeq} \text{ superposition of states of the form}$$
$$a^*(h_{1,t}) \ldots a^*(h_{n,t}) e^{-iHt}\psi_f$$

Asymptotic completeness of Compton scattering.

More precisely, AC for Compton of scattering is the statement that $\mathcal{H} = \mathcal{H}_+$ where

$$\mathcal{H}_+ = \overline{\mathrm{span}}\{a_+^*(h_1)\dots a_+^*(h_n)\varphi_f \,|\, h_i \in L^2_\omega,\ n \in \mathbb{N},\ f \in L^2\}.$$

The asymptotic creation and annihilation operators exist and the dressed one-electron wave packets φ_f are vacua for them.

Theorem. (FGS 2003) If

$$(a) \quad \Omega(P) = \sqrt{P^2 + M^2} \quad \text{and} \quad \Sigma < 3M/\sqrt{8} \quad \text{or}$$

$$(b) \quad \Omega(P) = P^2/(2M) \quad \text{and} \quad \Sigma < M/18,$$

then, for $|g|$ small enough, depending on Σ

$$\mathrm{Ran}\, E_\Sigma(H_g(P)) \subset \mathcal{H}_+.$$

Remark. The condition on Σ is chosen such that $|\nabla\Omega(P)| < 1/3$ for all P with $\Omega(P) \leq \Sigma$. This allows for electrons with speed as high as 10^8 m/s.

RELATED OPEN PROBLEMS

AC for Rayleigh scattering without IR cutoff:

- Show that the representation of the CCR by asymptotic fields is of Fock type. (easier than AC?)
- Prove a form of relaxation to the ground (easier than AC?).

- Combine Rayleigh and Compton scattering.
- Bohr frequency condition for small α.
- Photo effect.
- ...