

ELEMENTS OF SINGLE VARIABLE CALCULUS

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Functions. A function $f : A \rightarrow \mathbb{R}$ is a rule that assigns to each number x in the domain A , a number $f(x)$ in \mathbb{R} . f is the name of the function, $f(x)$ the value of the function at x . Ex.: $f = \sin$, $f(x) = \sin(x)$. Often A is understood to be the largest set where the given expression for f makes sense, and therefore not mentioned. The range B of f is the set of possible values $f(x)$ for x in the domain A . Ex.: The domain A of $f(x) = \sqrt{1-x^2}$ is the interval $[-1, 1]$, and the range B is the interval $[0, 1]$. The graph of f is the set of points $\{(x, f(x)) | x \in A\}$ in the xy -plane.

Inverse function. If f is one-to-one, that is, it takes no value twice, then one can define the inverse function $f^{-1} : B \rightarrow A$ by

$$x = f^{-1}(y) \iff y = f(x).$$

Ex.: If we choose $A = [-\pi/2, \pi/2]$ for the domain of $f = \sin$, then f is one-to-one, the range is $B = [-1, 1]$, and $f^{-1} = \arcsin$. *Caution:* Don't confuse $f^{-1}(x)$ with $f(x)^{-1}$ which is $1/f(x)$.

Exponential function and logarithm. The (natural) logarithm is the inverse of the exponential function. That is, for all $x \in \mathbb{R}$ and all $y > 0$:

$$(1) \quad \ln(e^x) = x, \quad e^{\ln y} = y.$$

For all $a, b, c \in \mathbb{R}$ and all $\alpha, \beta > 0$

$$\begin{aligned} e^{a+b} &= e^a e^b, & \ln(\alpha\beta) &= \ln(\alpha) + \ln(\beta) \\ e^{a-b} &= e^a / e^b, & \ln(\alpha/\beta) &= \ln(\alpha) - \ln(\beta) \\ (e^a)^b &= e^{ab}, & \ln(\alpha^c) &= c \ln(\alpha) \end{aligned}$$

The properties of \ln on the right follow from those of e^x by setting $\alpha = e^a$, $\beta = e^b$ and using (1).

For all $x, y > 0$ and all $a, b \in \mathbb{R}$

$$\begin{aligned} x^{a+b} &= x^a x^b, & (x^a)^b &= x^{ab} \\ x^{a-b} &= \frac{x^a}{x^b}, & (xy)^a &= x^a y^a. \end{aligned}$$

If a, b are integers, these equations are also true when x or y is negative.

Continuity. A function f is *continuous at* $x = a$ if

$$\lim_{x \rightarrow a} f(x) = f(a).$$

f is said to be *continuous* if it is continuous at every point of its domain.

Differentiation. The *derivative* $f'(x)$ of f at x is defined by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

provided this limit exists. Geometrically, $f'(x)$ is the slope of the graph of f at the point $(x, f(x))$ in the xy -plane. If $f'(x)$ exists for all x in the domain of f then f is called *differentiable*, and the assignment $x \mapsto f'(x)$ defines a new function f' called *derivative* of f .

Differentiation rules.

$$(f \pm g)' = f' \pm g', \quad (cf)' = cf'$$

$$(fg)' = f'g + fg'$$

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Mean value theorem. If $f : [a, b] \rightarrow \mathbb{R}$ is differentiable, then there exists a number $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

In particular, if $f'(x) = 0$ for all x then f must be constant.

Antiderivative. An *antiderivative* of f is a function F with $F' = f$. If F_1 and F_2 are two antiderivatives of f , then $F_1 - F_2$ is constant by the mean value theorem. The *indefinite integral*

$$(2) \quad \int f(x) dx$$

is a convenient notation for the family of all antiderivatives of a function f . That is, $\int f(x) dx = F(x) + c$, where c is a constant and $F' = f$. The process of finding F , given f , is called *integration*.

Integration rules.

$$\int (f + g) dx = \int f dx + \int g dx$$

$$\int cf dx = c \int f dx$$

$$\int f'g dx = fg - \int fg' dx$$

$$\int f(g(x))g'(x) dx = \int f(u) du \Big|_{u=g(x)}$$

Definite Integral. The *definite integral* from a to b of a continuous function f is a number defined by

$$\int_a^b f(x) dx := \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

where $\Delta x = (b - a)/n$ and $x_i = a + i\Delta x$. If $f(x) \geq 0$ on the interval $[a, b]$, then $\int_a^b f(x) dx$ is the area of the region in the xy -plane above the interval $[a, b]$ and below the graph of f .

Fundamental Theorem of Calculus. (a) If f is a continuous function defined on an interval I which contains the point a , then

$$\frac{d}{dx} \int_a^x f(t) dt = f(x) \quad \text{for all } x \in I.$$

That is, the function F , defined by $F(x) = \int_a^x f(t) dt$ is an antiderivative of f .

(b) If F is any antiderivative of f , then

$$F(b) - F(a) = \int_a^b f(x) dx.$$

Remarks. Part (a) of this theorem can be stated as

$$\int f(x) dx = \int_a^x f(t) dt + c,$$

thus justifying the notation (2) for the family of all antiderivatives. Part (b) follows from part (a).

Trigonometric identities.

$$(\sin \alpha)^2 + (\cos \alpha)^2 = 1$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(-\alpha) = -\sin \alpha$$

$$\cos(-\alpha) = \cos \alpha$$

$$(\sin \alpha)^2 = \frac{1}{2}(1 - \cos 2\alpha)$$

$$(\cos \alpha)^2 = \frac{1}{2}(1 + \cos 2\alpha)$$

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha}, \quad \cot x = \frac{1}{\tan x}$$

$$\csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}$$

Derivatives. Read the following table from left to right to find derivatives, read it from right to left to find antiderivatives.

$f(x)$	$f'(x)$
c	0
x^a	ax^{a-1}
$\frac{1}{a+1}x^{a+1}$	x^a
e^x	e^x
a^x	$a^x \ln a$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\ln(x)$	$\frac{1}{x}$

References. For more information and examples see e.g. the book *Calculus* by James Steward (2nd edition). Most relevant for this survey are the sections: 1.1 Functions, 1.6 Inverse Functions and Logarithms, 2.4 Continuity, 2.7 Derivatives, 2.8 The Derivative as a Function, 3.1 Derivatives of Polynomials..., 3.2 The Product and Quotient Rules, 3.3 The Chain Rule, 4.3 Derivatives and shapes of curves, 4.9 Antiderivatives, 5.2 The Definite Integral, 5.4 The Fundamental Theorem of Calculus, 5.5 The Substitution Rule, 5.6 Integration by Parts.

What is missing. Taylor's theorem, l'Hospital's rule, the intermediate value theorem and many other important results are missing in this survey of single variable calculus.