Complete strictly locally convex spacelike hypersurfaces in De Sitter space

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We are interested in finding complete spacelike (i.e. the induced metric is Riemannian) strictly locally convex immersions with constant curvature and with prescribed (compact) future asymptotic boundary Γ in De Sitter space dS_{n+1}. The so called steady state space $\mathcal{H}$, which features prominently in the cosmological models of Bondi-Gold and Hoyle, is only half of de Sitter space so is incomplete. Its boundary as a subset of dS_{n+1} is a null hypersurface which represents the past infinity of $\mathcal{H}$. The steady state space is isometric (with time orientation reversed) to $\mathbb{R}^n \times \mathbb{R}_+$ if $\mathbb{R}^{n+1}_+$ is endowed with the Lorentz metric

$$g_{(x,x_{n+1})} = \frac{1}{x_{n+1}^2}(dx^2 - dx_{n+1}^2), \tag{0.1}$$

which is called the half space model for $\mathcal{H}^{n+1}$. Because Σ is strictly locally convex and strictly spacelike, we are forced to take $\Gamma = \partial\Omega$ where $\Omega \subset \mathbb{R}^n$ is a smooth domain and seek Σ as the graph of a "spacelike" function $u(x)$ over Ω, i.e. $\Sigma = \{(x, x_{n+1}) : x \in \Omega,\ x_{n+1} = u(x), |\nabla u| < 1, \text{ in } \overline{\Omega}\}$. One easily computes that Σ is locally strictly convex in the half-space model if and only if $x^2 - u^2$ is (Euclidean) locally strictly convex . The asymptotic Plateau problem is then to find Σ satisfying

$$f(\kappa[\Sigma]) = \sigma > 1,\ \partial\Sigma = \Gamma\ .$$

where $\kappa[\Sigma] = (\kappa_1, \dots, \kappa_n)$ denote the (positive) principal curvatures of Σ. I will describe recent joint work with my student Ling Xiao in which we solve this problem in great generality.