CONFORMAL GEOMETRY SEMINAR
The Poincaré Uniformization Theorem

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We will assume that all the manifolds M are compact and orientable unless otherwise stated. In this first part of the seminar we will prove the Poincaré Uniformization Theorem.

Theorem 1 (Poincaré Uniformization Theorem). *Let (M, g) be a compact 2-dimensional Riemannian manifold. Then there is a metric $\widetilde{g} = e^{2u} g$ conformal to g which has constant Gauss curvature constant.*

1. Preliminaries

1.1. **Geometry.** A covariant derivative on a manifold M is an operator $\nabla_X Y$ on vector fields X and Y satisfying for any smooth function f: (i) $\nabla_{fX} Y = f \nabla_X Y$; and (ii) $\nabla_X (fY) = f \nabla_X Y + (\nabla_X f) Y$. If g is a Riemannian metric on M, then there is associated with g a unique covariant derivative ∇ characterized by: (iii) $\nabla_X Y - \nabla_Y X = [X, Y]$; and (iv) $\nabla_X \big(g(Y, Z)\big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$. We define the Christoffel symbols by $\Gamma^i_{jk} = dx^i \big(\nabla_{\partial_j} \partial_k\big)$, where ∂_i is a coordinate basis, and dx^i is the dual basis. The Christoffel symbols can be computed from:

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} (\partial_j g_{mk} + \partial_k g_{mj} - \partial_m g_{jk}).$$

A curve γ is a geodesic if $\nabla_{\dot\gamma} \dot\gamma = 0$. Geodesics locally minimize arclength $\int_\gamma |\dot\gamma|$. A Riemannian manifold is complete if there are no inextendible geodesics. In a complete Riemannian manifold, any two points can be joined by a length minimizing geodesic. If M is a complete Riemannian manifold, and $x \in M$, the map $T_x M \to M$ sending each $X \in T_x M$ to the point $\gamma(1)$, where γ is the geodesic with $\gamma(0) = x$ and $\dot\gamma(0) = X$, is denoted $\exp_x$, and is called the exponential map at x. The radius of injectivity i_x at x is the supremum over all $R > 0$ such that $\exp_x$ is non-singular on $B_R(0) \subset T_x M$.

The Riemann curvature operator is defined by:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The components of the Riemann curvature operator, or curvature tensor, are given in a coordinate basis by:

$$R_{ijkl} = g(R(\partial_k, \partial_l)\partial_j, \partial_i),$$
$$R^i{}_{jkl} = g^{im} R_{mjkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{km} \Gamma^m_{jl} - \Gamma^i_{lm} \Gamma^m_{jk},$$

The curvature tensor has the following symmetries:

$$R_{ijkl} = -R_{ijlk} = -R_{jikl} = R_{klij}, \qquad R_{ijkl} + R_{iljk} + R_{iklj} = 0.$$

The Ricci curvature tensor is the trace of the Riemann tensor:

$$R_{ij} = g^{kl} R_{ikjl}.$$

The scalar curvature is the trace of the Ricci curvature:

$$R = g^{ij} R_{ij}.$$

The Laplace operator of g is:

$$\Delta_g u = g^{ij}\nabla_i\nabla_j u = \frac{1}{\sqrt{\det g}}\partial_i(g^{ij}\sqrt{\det g}\,\partial_j u).$$

Under a conformal transformation $g \mapsto \widetilde{g} = e^{2u}g$, the scalar curvature changes according to:

$$R \mapsto \widetilde{R} = e^{-2u}(R - 2(n-1)\Delta_g u - (n-1)(n-2)\,|\nabla u|^2). \tag{1}$$

A important special case is scaling, i.e.,u constant, say $e^{2u} = \lambda$, in which case $R \mapsto \widetilde{R} = \lambda^{-1}R$. If Σ is a hypersurface in M, then g induces a Riemannian metric g' on Σ. The second fundamental form h of Σ is defined by:

$$h(X,Y) = -g(\nabla_X N, Y), \qquad h_{ij} = -\nabla_i N_j$$

where N is the outer unit normal to Σ, and X, Y are tangent to Σ. It is not difficult to see that h is a symmetric tensor on Σ. We have the Gauss and Codazzi equations:

$$R'_{ijkl} = R_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}, \tag{2}$$

$$\nabla_k h_{ij} - \nabla_j h_{ik} = R_{lijk}N^l, \tag{3}$$

where R'_{ijkl} is the Riemann curvature tensor of g'.

When $n = 2$, we have:

$$R_{ijkl} = \kappa(g_{ik}g_{jl} - g_{il}g_{jk}), \qquad R_{ij} = \kappa g_{ij}, \qquad R = 2\kappa,$$

where κ is the Gauss curvature. From (1), we now get:

$$\widetilde{\kappa} = e^{-2u}(\kappa - \Delta_g u). \tag{4}$$

If γ is a curve in M, parametrized by arclength $|\dot{\gamma}| = 1$, then we can choose the normal N so that $(\dot{\gamma}, N)$ is positively oriented. Note that $\ddot{\gamma} = hN$, since $g(\ddot{\gamma}, N) = 0$; h is the geodesic curvature of γ. If γ is the boundary curve of a smooth domain Ω, we parameterize γ so that N points into the exterior of Ω.

Theorem 2 (Gauss-Bonnet I)**.** *Let Ω be a bounded smooth simply connected domain in a 2-dimensional oriented Riemannian manifold. Let γ be the boundary of Ω, and let h be the geodesic curvature of γ. Then, we have:*

$$\int_\Omega \kappa = 2\pi - \int_\gamma h.$$

Theorem 3 (Gauss-Bonnet II)**.** *Let (M,g) be a compact 2-dimensional Riemannian manifold, and let χ be its Euler characteristic. Then, we have:*

$$\int_M \kappa = 2\pi\chi. \tag{5}$$

Remark: χ is a topological invariant. In fact $\chi = \sum_{j=0}^n (-1)^j b_j$ where b_j is the j-th Betti number of M.

1.2. **Analysis.** The equation:

$$\Delta_g u = f$$

has a solution, unique up to a constant, if and only if f has zero mean, i.e.,

$$\bar{f} = \fint f = \frac{1}{\mathrm{vol}(M)} \int_M f = 0.$$

More precisely, let $W_k^p(M)$ be the Sobolev space of functions u on M such that:

$$\|u\|_{k,p}^p = \sum_{j=0}^{k} \int_M \left|\nabla^j u\right|^p < \infty, \tag{6}$$

and let $W_{k,0}^p(M) \subset W_k^p(M)$ be the subspace of those u with zero mean. The derivatives are meant in the weak sense, so that if $u, f \in L^p(M)$, then $\Delta_g u = f$ means:

$$\int_M u \, \Delta_g \varphi = \int_M f\varphi,$$

for all smooth test functions $\varphi \in C^\infty(M)$. These spaces with the norms (6) are Banach spaces. When $p = 2$, they are Hilbert spaces, and we write $W_k^2(M) = H_k(M)$, and $W_{k,0}^2(M) = H_{k,0}(M)$.

Theorem 4 (Sobolev Embeddings)**.** *Let (M, g) be a compact Riemannian n-dimensional manifold. Then:*

(i) *If $p < n$, then $W_1^p(M) \hookrightarrow L^q(M)$ for all $q \leq np/(n-p)$. If $q < np/(n-p)$, then the embedding is compact.*
(ii) *If $p > n$, then $W_1^p(M) \hookrightarrow C^\alpha(M)$ where $\alpha \leq 1 - n/p$. If $0 < \alpha < 1 - n/p$, then the embedding is compact.*
(iii) *If $p = n$ and $u \in W_1^n(M)$, then $e^u \in L^1(M)$, and $\log\left(\int_M e^u\right) \leq C(1 + \|u\|_{1,n}^n)$. Furthermore, the map $W_1^n(M) \hookrightarrow L^1(M)$ given by $u \mapsto e^u$ is compact.*

Theorem 5 (Poincaré Inequality)**.** *Let $\lambda_1 > 0$ be the first positive eigenvalue of $-\Delta_g$. Then*

$$\int_M u^2 \leq \frac{1}{\lambda_1} \int_M |\nabla u|^2 , \tag{7}$$

for all $u \in H_{1,0}(M)$.

Proof. Let $\mu \geq 0$ be the infimum of $\int_M |\nabla v|^2$ over the set S of all $v \in H_{1,0}(M)$ such that $\int_M v^2 = 1$. Let v_i be a minimizing sequence, then v_i is bounded in $H_{1,0}(M)$, hence it has subsequence, say v_i again, which converges weakly to $v \in H_1(M)$. By Theorem 4, we may assume, by passing again to a subsequence, that v_i converges strongly in $L^2(M)$, hence also strongly in $L^1(M)$, to v, and thus $\bar{v} = 0$, and $\int_M v^2 = 1$, i.e.,$v \in S$. Now, $v \mapsto \int_M |\nabla v|^2$ is weakly lower semi-continuous on $H_1(M)$. Indeed,

$$\int_M |\nabla v|^2 = \lim_i \int_M g(\nabla v_i, \nabla v) \leq \liminf_i \left(\int_M |\nabla v_i|^2\right)^{1/2} \left(\int_M |\nabla v|^2\right)^{1/2} = \mu^{1/2} \left(\int_M |\nabla v|^2\right)^{1/2}.$$

Canceling, and squaring, we get that $\int_M |\nabla v|^2 \leq \mu$, hence v is a minimizer in S. Note that $\mu > 0$, for otherwise, v would be constant, hence zero since $\bar{v} = 0$, in contradiction to $v \in S$. If $u \in H_{1,0}(M)$, then $\left(\int_M u^2\right)^{-1/2} u \in S$, and (7) follows with λ_1 replaced by μ. To show that $\mu = \lambda_1$, note that for some $\lambda \in \mathbb{R}$:

$$\int_M g(\nabla v, \nabla \varphi) - \lambda v \varphi = 0, \qquad \forall \varphi \in C_0^\infty(M).$$

Since $\bar{v} = 0$, this holds also for all $\varphi \in C^\infty(M)$, and by density for all $\varphi \in H_{1,0}(M)$. This is the weak form of $-\Delta_g v = \lambda g$. The value of λ is found by setting $\varphi = v$. Regularity of v follows as in Theorem 6. Thus, μ is an eigenvalue of $-\Delta_g$, hence $\mu \geq \lambda_1$. Let $u_1 \in S$ be an eigenfunction of $-\Delta_g$ with eigenvalue λ_1, $-\Delta_g u_1 = \lambda_1 u_1$. Mutliplying by u_1 and integrating by parts, we get that $\int_M |\nabla u_1|^2 = \lambda_1$, hence $\lambda_1 \geq \mu$. This shows that $\mu = \lambda_1$. □

Theorem 6. *Let (M, g) be a compact Riemannian manifold. Then, for all $k \geq 0$, and $1 < p < \infty$, the operator $\Delta_g \colon W^p_{k+2,0}(M) \to W^p_{k,0}(M)$ is an isomorphism*

Proof. We will only sketch part of the proof for $p = 2$, $k = 0$, i.e.,we show that Δ_g is an isomorphism $H_{2,0}(M) \to L^2_0(M)$. To prove injectivity, assume $\Delta_g u = 0$, multiply by u, and integrate by parts:

$$\int_M |\nabla u|^2 \leq 0.$$

Hence u is constant, and since $\bar{u} = 0$, it follows that $u = 0$. Continuity is straightforward. Continuity of the inverse will follow from the inequality

$$\|u\|_{2,2} \leq C\|\Delta_g u\|_{0,2}. \tag{8}$$

By a density argument, it suffices to prove (8) for $u \in C^\infty_0(M)$. Write $\Delta_g u = f$. Multiply by u, and integrate by parts to get:

$$\int_M |\nabla u|^2 = -\int_M fu \leq \frac{\epsilon}{2}\int_M u^2 + \frac{1}{2\epsilon}\int_M f^2,$$

for any $\epsilon > 0$. Pick $\epsilon = 1/\lambda_1$ where λ_1 is the first positive eigenvalue of $-\Delta_g$, and use the Poincaré inequality, Theorem 5, to get:

$$\|\nabla u\|_{1,2} \leq \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_1^2}\right)^{1/2} \|f\|_{0,2}.$$

It remains to estimate the second derivatives in L^2. Differentiate $\Delta_g u = f$:

$$\Delta_g \nabla_i u = \nabla_i f + R_{ij}\nabla^j u,$$

and take inner product with ∇u:

$$g(\nabla u, \Delta_g \nabla u) = g(\nabla u, \nabla f) + \mathrm{Ric}(\nabla u, \nabla u).$$

Integrating over M gives:

$$\int_M |\nabla^2 u|^2 = \int_M f^2 - \int_M \mathrm{Ric}(\nabla u, \nabla u) \leq \|f\|^2_{0,2} + \Lambda\|\nabla u\|^2_{0,2},$$

where $\Lambda = \max\{-\min_M \lambda_{\mathrm{Ric}}, 0\}$, and λ_{Ric} is the smallest eigenvalue of Ric with respect to g. This completes the proof of (8). Note that this prove that the range R of Δ_g is closed in $L^2_0(M)$. To show that R is also dense in $L^2_0(M)$, assume now that $f \in L^2_0(M)$ is orthogonal R, then

$$\int_M f\Delta_g \varphi = 0, \qquad \forall \varphi \in C^\infty_0(M)$$

This implies that $f \in H_{1,0}(M)$. Integrating by parts, the same holds for all $\varphi \in H_{1,0}(M)$ by density. Substitute $\varphi = f$ to show that $f = 0$. It follows that $R = L^2_0(M)$. The proof for $p = 2$, and $k \geq 1$ is similar. The proof for $p \neq 1$ requires estimates on the fundamental solution. □

If $u \in C^k(M)$, we define

$$\|u\|_k = \sum_{j=0}^{k} \sup_M \left|\nabla^j u\right|.$$

If T is a Hölder continuous tensor on M, we define:

$$\|T\|_\alpha = \sup_{x \neq y} \frac{\left|P_y^x T(y) - T(x)\right|}{\operatorname{dist}(x,y)^\alpha},$$

where the supremum is taken over all $x \neq y$ in some convex neighborhood, and P_y^x is parallel translation along the unique geodesic from y to x. The space $C^{k,\alpha}(M)$ is the space of functions $u \in C^k(M)$ such that $\|\nabla^k u\|_\alpha < \infty$. The norm on $C^{k,\alpha}(M)$ is defined to be:

$$\|u\|_k + \|\nabla^k u\|_\alpha.$$

The subspace of those functions $u \in C^{k,\alpha}(M)$ with mean zero is denoted $C_0^{k,\alpha}(M)$.

Theorem 7. *Let (M,g) be a compact Riemannian manifold. Then, for all $k \geq 0$, and $0 < \alpha < 1$, the operator $\Delta_g \colon C_0^{k+2,\alpha}(M) \to C_0^{k,\alpha}(M)$ is an isomorphism*

2. The Poincaré Uniformization Theorem

In view of the Gauss-Bonnet Theorem II, the proof is divided into three cases. Indeed, by (5), the sign of $\widetilde{\kappa}$ is determined by the topology of M.

2.1. Zero Euler Characteristic.

In this case, we will conformally transform g to a metric of zero Gauss curvature. In view of (4), we must find u such that

$$\Delta_g u = \kappa.$$

Since $\int_M \kappa = 0$, this equation has a unique solution with zero mean by Theorem 7. Furthermore, since $\kappa \in C^\infty(M)$, we have from Theorem 7 that $u \in C^\infty(M)$. This completes the proof in the case of zero Euler characteristic.

2.2. Negative Euler Characteristic.

In this case, we will conformally transform g to a metric $\widetilde{g}$ of Gauss curvature $\widetilde{\kappa} = -1$. In view of (4), we must solve:

$$\Delta_g u - e^{2u} = \kappa. \tag{9}$$

Note that the solution is now unique for if u and v are both solutions of (9), then

$$\Delta_g(u - v) = e^{2v}(e^{2(u-v)} - 1).$$

Multiplying by $u - v$ and integrating by parts, we get: $\int_M |\nabla(u-v)|^2 \leq 0$, i.e.,$u - v$ is constant, which implies that $u = v$.

Since under a constant scaling $g \mapsto \lambda g$, we have $\kappa \mapsto \lambda^{-1}\kappa$, and $\bar{\kappa} \mapsto \lambda^{-1}\bar{\kappa}$, we may assume, without loss of generality, that $\bar{\kappa} = -1$. There is a unique function $v \in C^\infty(M)$ such that $\bar{v} = 0$, and $\Delta_g v = \kappa + 1$. Let $u = v + w$, then we must solve:

$$\Delta_g w - e^{2v} e^{2w} = -1. \tag{10}$$

We will minimize:

$$F(w) = \int_M \left(\frac{1}{2}|\nabla w|^2 - w\right),$$

over the set of S of functions $w \in H_1(M)$, such that @!@!@!@! $\int e^{2v} e^{2w} = 1$.

If w is a minimizer, then by Lagrange multipliers, we have:

$$\int_M \left(g(\nabla w, \nabla \varphi) - \varphi + \lambda e^{2v} e^{2w} \varphi \right) = 0,$$

for all smooth test functions $\varphi \in C^\infty(M)$. Taking $\varphi = 1$, yields $\lambda = 1$. Thus, w satisfies (10) weakly. By Theorem 4 (iii), $e^{2w} \in L^p(M)$ for all $p \geq 1$, thus $\Delta_g w \in L^p(M)$ for all $p \geq 1$. Pick $p > 2$, then by Theorem 6, we get that $w \in W_2^p(M)$, and hence by Theorem 4, $w \in C^{1,\alpha}(M)$, $\alpha = 1 - 2/p$. Thus, $\Delta_g w \in C^{1,\alpha}(M)$, and we now get $w \in C^{3,\alpha}(M)$. By induction, we conclude that $w \in C^\infty(M)$.

It remains to show that F has a minimizer in S. Let $\mu = \inf_S F$. By Young's inequality:

$$⨍ (2v + 2w) \leq \log ⨍ e^{2v+2w} = 0,$$

Thus $\int w \leq 0$, and $\mu \geq 0$. Let w_j be a minimizing sequence. Then $\|\nabla w_j\|_{0,2} \leq C$, and $-C \leq \bar{w}_j \leq 0$, for some constant C. It follows, using Theorem 5, that:

$$\|w_j\|_{0,2} \leq \|\bar{w}_j\|_{0,2} + \|w - \bar{w}_j\|_{0,2} \leq C \operatorname{vol}(M)^{1/2} + \lambda_1^{1/2} \|\nabla w\|_{0,2} \leq C'.$$

Thus, w_j is bounded in $H_1(M)$, and has a subsequence which converges weakly to w. Using Theorem 4 (i) and (iii), we may assume, by passing to a subsequence, that w_j and e^{2w_j} converge strongly in $L^1(M)$ to w and e^{2w} respectively. It follows that @!@!@!@! $\int e^{2v} e^{2w} = 1$, and hence $w \in S$. Furthermore, as in the proof of Theorem 5, F is weakly lower semi-continuous on $H_1(M)$. Thus, $F(w) \leq \liminf F(w_j)$. We conclude that $F(w) = \mu$, and w is a minimizer. This completes the proof in the case of negative Euler characteristic.

2.3. **Positive Euler Characteristic.** Note that since $b_0 = b_2 = 1$, we get $\chi \leq 2$, and since M is orientable, b_1 is even. Indeed, by the Hodge Decomposition Theorem, each cohomology class has a unique harmonic generator. If ω is a harmonic generator of the first cohomology of M, and $*$ is the Hodge dual, then $*\omega$ is also harmonic, and perpendicular to ω, hence it is another generator, i.e.,generators of $H_1(M)$ come in pairs. Thus, we assume that $\chi = 2$. In this case, we will conformally transform g to a metric $\widetilde{g} = e^{2u} g$ of Gauss curvature $\widetilde{\kappa} = 1$. In view of (4), we must solve:

$$\Delta_g u + e^{2u} = \kappa. \tag{11}$$

Note that the solution is not unique. Indeed, assume that $M = \mathbb{S}^2$, and $g = g_0$ the standard metric. Then (11) becomes:

$$\Delta_g u + e^{2u} = 1. \tag{12}$$

If $\widetilde{g} = e^{2u} g$ is isometric to g, then u satisfies (12). A function $\phi \colon M \to M$ is called a conformal isometry if ϕ is an isometry and $\phi_* g = e^{2u} g$ pointwise. The group of conformal isometries of $(\mathbb{S}^2, g)$ is the Lorentz group, a 6-parameter non-compact group. It follows that (12) has many solutions, and in fact, arbitrarily large ones. To see this, write g in stereographic coordinates:

$$ds^2 = \frac{|dz|^2}{\left(1 + \frac{1}{4}|z|^2\right)^2}.$$

The conformal isometries are the bi-linear transformations:

$$f(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad \alpha\delta - \beta\gamma = 1.$$

The conformal factor is given by:

$$e^{2u} = \frac{|f'(z)|^2 \left(1 + \frac{1}{4}|z|^2\right)^2}{\left(1 + \frac{1}{4}|f(z)|^2\right)^2}.$$

With the scaling $f(z) = \alpha z$, $\alpha \in \mathbb{R}$, we get:

$$e^{2u} = \frac{\alpha^2 \left(1 + \frac{1}{4}|z|^2\right)^2}{\left(1 + \frac{1}{4}\alpha^2 |z|^2\right)^2}.$$

In particular, there are no a priori estimates for solutions of (11). We conclude that the method of continuity cannot work.

The variational method also fails. As before, we can find $v \in C^\infty(M)$ which satisfies $\bar{v} = 0$, and $\Delta_g v = \kappa$. Substituting $u = v + w$, to obtain:

$$\Delta_g w + e^{2v+2w} = 1.$$

However, the functional:

$$F(w) = \int_M \left(\frac{1}{2}|\nabla w|^2 + w\right),$$

has no lower bound on the set S of functions $w \in H_1(M)$ such that @!@!@!@!$\int e^{2v}e^{2w} = 1$.

Instead, our approach will be the following. We will find a function v on $M \setminus \{x\}$ such that $\Delta_g v = \kappa$, i.e.,$e^{2v}g$ has zero Gauss curvature, and we will then show that $\left(M \setminus \{p\}, e^{2v}g\right)$ is isometric to the Euclidean plane. Now, clearly there is a function w on $M \setminus \{x\}$ such that $\left(M \setminus \{x\}, e^{2w}e^{2v}g\right)$ is isometric to $\left(\mathbb{S}^2 \setminus \{p\}, g_0\right)$. The last step will be to show that $u = v + w$ extends smoothly to M.

Pick $\epsilon > 0$ so that $2\epsilon < \iota_x$ radius of injectivity of $\exp_x$. Let $\varphi \in C^\infty(M)$ such that $\varphi = 1$ on $B_\epsilon(x)$, and $\varphi = 0$ on $M \setminus B_{2\epsilon}(x)$. Let s be the geodesic distance from x, and define $f = \kappa + 2\Delta_g(\varphi \log s)$. We will now show that $f \in L^\infty(M)$, and @!@!@!@!$\int f = 0$. Clearly, f is continuous on $M \setminus B_\epsilon(x)$.

Introduce in $B_{2\epsilon}(x)$ normal polar coordinates (s, θ) centered at x:

$$g_{ij}\, dx^i\, dx^j = ds^2 + r^2(s,\theta)d\theta^2, \tag{13}$$

Lemma 1. *Let g be a smooth metric on a 2-dimensional geodesic ball $B_R(x)$ of radius $R \leq i_x$ centered at x, and let (s, θ) be normal polar coordinates centered at x, so that the line element of g is* (13)*. Then $r = s(1 + s^2\xi)$, where $\xi \in C^\infty(B(x))$.*

Proof. Clearly $r = O(s)$, hence it suffices to prove that $dr/ds(0) = 1$, and $d^2r/ds^2(0) = 0$ along any radial geodesic from x. Let $h = r^{-1}dr/ds$ be the logarithmic expansion of the geodesic circles $s =$ constant; h is the 'mean' curvature of the geodesic circles and is also the geodesic curvature of these circles. Thus, the Gauss-Bonnet Theorem I gives:

$$\int_{B_s(x)} \kappa = 2\pi - \int_0^{2\pi} hr\, d\theta = 2\pi - \int_0^{2\pi} \frac{dr}{ds} d\theta,$$

which implies immediately that $dr/ds(0) = 1$. Furthermore, along any radial geodesic, we have:

$$\frac{dh}{ds} = -h^2 - \kappa. \tag{14}$$

Contract the Codazzi equation (3) with respect to i and j:

$$\nabla_k \operatorname{tr} h - \nabla^j h_{jk} = -R_{lk}N^l,$$

and contract again with the normal N^k to get:

$$\nabla_N \operatorname{tr} h - N^k \nabla^j h_{jk} = -\operatorname{Ric}(N, N).$$

Now, to obtain (14), note that $N^k \nabla^j h_{jk} = -\nabla^j N^k h_{jk} = -h^{jk} h_{jk} = -|h|^2$. Substituting $h = r^{-1}\partial r/\partial s$ in (14), we now get:

$$\frac{d^2 r}{ds^2} = \kappa r. \tag{15}$$

Clearly, if r is a solution of (15), with $r(0) = 0$, we get $d^2r/ds^2(0) = 0$. $\square$

We have:

$$\Delta_g(\log s) = \frac{1}{r}\frac{\partial}{\partial s}\left(\frac{r}{s}\right),$$

which, in view of Lemma 1 is smooth. Since $r/s = 1 + O(s^2)$, we have

$$\int_{M \setminus B_\delta(x)} \Delta_g(\varphi \log s) = -\int_0^{2\pi} 2\frac{\partial}{\partial s}(\log s) r\, d\theta = -\int_0^{2\pi} 2\,\frac{r}{s}\, d\theta \to -4\pi,$$

as $\delta \to 0$. Hence taking the limit $\delta \to 0$, and noting that $\int_M \kappa = 4\pi$, we obtain $\int_M f = 0$, as claimed. It now follows from Theorem 7 that there is a unique smooth function $\hat{v}$ on M such that $\Delta_g \hat{v} = f$, and $\hat{v}(x) = 0$. Set:

$$v = -2\varphi \log s + \hat{v}, \qquad g' = e^{2v} g.$$

Then $\Delta_g v = \kappa$ on $M' = M \setminus \{x\}$, and thus (M', g') has constant Gauss curvature $\kappa' = 0$, i.e.,is flat. It is also easy to see that (M', g') is complete. Indeed, setting $t = 1/s$, the metric g' can be written as:

$$\left(dt^2 + (t^2 + \xi) d\theta^2\right)\left(1 + O(1/t)\right), \tag{16}$$

where ξ is bounded, hence any curve $\gamma(t) = \big(s(t), \theta(t)\big)$ through x has infinite length in the metric g'. The universal cover $(\hat{M}, \hat{g})$ of M' is a simply connected complete flat surface, hence isomorphic to the Euclidean plane. Thus (M', g') is the quotient of the plane by a totally discontinuous group of isometries. The only totally discontinuous subgroups of the isometry group of the plane are generated by translations. If the subgroup had two generators, then M' would be a torus, which is impossible, since M' is not compact. If the subgroup had one generator, then M' would be a cylinder, which is impossible, since it cannot be compactified by adding one point. Thus (M', g') is the flat plane.

Fix a point $o \in M'$, and let ρ be the Euclidean distance from o. Define:

$$w = -\log\left(1 + \frac{\rho^2}{4}\right), \qquad u = v + w, \qquad \widetilde{g} = e^{2u} g = e^{2w} g'.$$

Then $\widetilde{g}$ has constant Gauss curvature $\widetilde{\kappa} = 1$ on M'. However, using (16), it is easily seen that $|\rho - t| = |\rho - 1/s|$ is bounded in $M \setminus B_\epsilon(x)$, hence $w = -2\log s + O(s)$, hence $u(x) = 0$, i.e.,$\widetilde{g}$ extends smoothly to M. This completes the proof of the Poincaré Uniformization Theorem.

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